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A Mathematical-Logic Technique Facilitating Good Teaching

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ABSTRACT

This paper presents a teaching strategy for topics in undergraduate courses which require simultaneous consideration of several parameters. Such topics present several challenges: i) initial learning is difficult because of the multiple issues (parameters) that must be checked, ii) covering all cases can leave students confused, leading to omission of certain details, iii) it is not clear what presentation vehicle is best for learning. This paper proposes a mathematical-logic technique, the prime implicant normal form, PINF, to address these problems. The PINF method can easily be mastered without technical knowledge. Familiar examples are provided from Statistics and English Grammar. A survey of alternate presentation methods, gleaned from a simple Google search, reveals several advantages of using the PINF method.

Keywords: prime implicant normal form, instructional design, good pedagogy, normal distribution, student t distribution, relative pronouns, matrix organizers, tables,

1. BACKGROUND and GOALS

Recall, that pedagogy theory teaches that identical content can be delivered by different vehicles, with some vehicles excelling over others in facilitating student learning and retention.

Perhaps the simplest illustration of a pedagogically superior presentation method is the use of graphical organizers such as tables or matrix displays [10, 11, 17]. Although a bulleted list can present *the same information* as the table, the table provides quicker *access* to contrasts and comparisons, because it displays more relationships. Hence, it is superior for presentation.

This paper presents a superior teaching method motivated by a technique of mathematical logic, the *prime-implicant normal form*, PINF, of Boolean algebra. This method is particularly useful when a course topic requires simultaneous consideration of multiple parameters. The capacity of PINF to facilitate student learning and retention is just as powerful as the use of graphic organizers but not as commonly known.

To illustrate the pedagogic superiority of this technique we select two undergraduate topics, one from statistics (Section 2) and the second from English grammar (Section 4). Contrastive to traditional textbooks and online teaching resources which present either by using bulleted lists, complicated flowcharts, or omitting cases (Section 3)., the PINF is comprehensive, more compact, and clearer. The paper concludes by advocating that instructors incorporate the PINF technique into their courses (Section 5).

2. NORMAL, t, or PARAMETRIC DISTRIBUTIONS

2.1 Background: Statistics deals with making valid inferences from a sample obtained from a population. Typically, a sample statistic is computed, for example, the average or proportion of the particular sample studied. Statistics then informs you how confident you can be that the corresponding population statistic is within a certain margin of error from the sample statistic. In making this inference, the analyst must make computations with certain standard distributions. The three most frequently used methods used in making inferences from a sample to a population are i) the normal distribution, ii) the student *t*-distribution, and iii) non-parametric methods. Therefore, the analyst needs a *decision rule* by which to select the appropriate method. This entire theory is well understood and presented in numerous textbooks which typically service introductory statistic courses [5, 18].

Using the PINF technique, Section 2.2 presents a decision rule for how to select the appropriate one of these three methods. Section 2.3 is easy to follow; Section 2.3 explains the approach of the underlying PINF technique used without requiring familiarity with technical mathematical-logic. Section 3 then compares the PINF technique advocated by this paper with several alternative presentation vehicles.

2.2 The Decision Rule: Table 1, constructed with the PINF technique, compactly presents the decision rule for which of the three methods to use. To arrive at a conclusion of which distribution to use (last column), the table requires simultaneous consideration of several parameters, or attributes, of the given sample and the underlying population.

Is the sample size large? (i.e., at least 30 (See Section 2.4))	Is the population normally distributed?	Is the population variance known?	Then use the following distribution
Yes			Normal
	Yes	Yes	Normal
No	Yes	No	Student t
No	No		Parametric
No		No	Parametric

Table 1: The PINF approach to choosing the correct method by which to make inferences from a sample.

Although Table 1 should be self-explanatory, we review the first two rows to fully clarify its meaning.

- Row 1: The first question to consider is whether the sample size is large (Column 1). If it is, no other questions need be asked; one should use the normal distribution (last column).

- Row 2: If the answer to the following two questions is, "Yes",
  - Is the population known to be normal? (Column 2)
  - Is the variance of the population known? (Column 3),

then we use the normal distribution to make inferences (last column).

2.3 PINF: To clarify the PINF technique used in Table 1, we contrast it with the *disjunctive normal form* taught in mathematic logic courses. These courses traditionally use a *truth table.* The truth table would contain one column for each question asked. In the example studied in this section we consider three questions, i) Is the sample large? ii) Is the population normal? iii) Is the variance of the population known? as shown in the first three columns of Table 1.

Notice that each of these three questions has a *yes* or *no* response. The disjunctive normal form requires that the *truth table* contains all eight possibilities of *yes and no,* each possibility presented in a separate row. For each row, the table then indicates whether the method is normal, t, or parametric.

The contrast to the disjunctive normal form introduced by the PINF table is that blank cells are allowed which contain neither *yes* nor *no*. We can illustrate this with the first row of Table 1. The disjunctive normal form would require the following four rows to replace the first row of Table 1.

Is the sample size large? (usually bigger than 30; but other numbers are possible)	Is the population normal?	Is the population variance known?	Then use the following distribution
Yes	No	No	Normal
Yes	No	Yes	Normal
Yes	Yes	No	Normal
Yes	Yes	Yes	Normal

Table 2: The first row of Table 1 would require the four rows of Table 2 if disjunctive normal form were used.

By comparing Table 2 with the first row of Table 1 we immediately see the advantage of PINF. The PINF table focuses on what drives use of the normal form, namely *largeness* of the sample. Contrastively, Table 2 is too busy; it overwhelms the reader, not allowing such a focus.

This example illustrates *how* an instructor can construct a PINF table: i) First gather all rules, ii) next list all questions, issues, or parameters that must be asked, iii) assure that each of these questions has simple *yes-no* responses, iv) list all known rules in a table, v) then, if not already done, reduce redundant rows and information in the table by leaving certain cells blank.

2.4 A Technical Comment.: Because the purpose of this paper is presentation of the PINF method, it abstains from making excessively technical statistical points. We illustrate such omissions with one subtlety. In an actual course such subtleties would, of course, be mentioned. Table 1 requires ascertaining whether a sample is *large.* Typically, textbooks will suffice with classifying a sample as *large* if it has 30 or more elements. The reason for this is that for samples of 30 or more elements, the student *t* and normal distribution are very close in numerical value; hence, using either one is appropriate.

But this argument is only valid when the desired significance level is the typical 5%, that is, when one wishes to be 95% confident that the sample is predictive of the population. For some applications, a 99% or even a 99.99% confidence level may be desired. For these applications, the cutoff point for classifying a sample as large would be bigger than 30. However, as remarked, this is a very technical point and need not concern us further in this paper.

3. ALTERNATIVE FORMULATIONS

To defend the superiority of the PINF teaching delivery method, alternate approaches to the decision rule for sampling were randomly obtained, in March 2023, by a simple Google search, using the search phrase "when to use the normal vs. *t* distribution" (including "parametric distribution" in the search phrase did not significantly change the search results). Several alternate approaches were obtained [2, 3, 7, 9, 13, 20], some from university websites, some from online tutorials, and some from expository articles in refereed journals; these sources were supplemented with two textbooks [5,18].

The alternate approaches were qualitatively inferior; they frequently omitted cases, omitted parameters, or presented the decision rule in stages leading to unintended contradictory statements of the decision rule.

Rather than review each source by itself, we summarize several repeating issues.

<u>Issue #1: A Contrastive Narrative Style:</u> A contrastive style is easy to retain but may omit information. Two examples of contrastive styles are the following:

(1) For large samples use normal; for small samples use *t*.
(2) When the variance is known use normal; if unknown use *t*.

Although (1) and (2) are useful rules of thumb, they are not complete. For example, both formulations leave out the critical information that one should not use the *t*-distribution unless the population is known to be normal.

<u>Issue #2: A Development Style</u>: Some narratives *develop* the complete set of decision rules over several pages. This development may begin with a contrastive style like (1) or (2) and then, in a later paragraph, remark that other requirements (such as normalcy of a population) are needed.

While such a development style is easier to read than a table, it presents multiple statements which, if read alone, are contradictory, the initial statement being too simple and the final statement being correct, albeit technical. This can easily confuse a student that has before him or her two formulations.

One simple remedy to this is to *supplement* narrative development styles with a summary such as Table 1. This has the advantage of providing a one-stop complete set of rules (the table) but developing the ideas at a leisurely pace which helps weaker students.

<u>Issue #3: Confusing Logical Connectives</u>: The *exportation* law of classical logic states that the following two formulations of

an abstract rule are equivalent.

(3) If both (A) and (B) are true then do (C)
(4) If (A) is true then we consider further cases. If (B) is true, then do (C).

Formulation (4) is appealing when presenting rules using a nested bulleted list. The main (outer) bullets could present *variance known* and *variance unknown*. The sub-bulllets of *variance unknown* could then present the cases of the *sample large* or *not large*.

But formulation (4) which fits in with a nested-bulleted-list presentation is harder to read than formulation (3). In fact, one point of the exportation law is that very often a reader will *understand* (4) by first reformulating it as (3).

An additional point in comparing nested bulleted lists with tables, is, as pointed out earlier, that tables are superior since they facilitate comparisons and contrasts [10, 11, 17].

<u>Issue #4: Lack of Completeness:</u> One should be careful in a critique of lack of completeness. A textbook or handout could, for example, argue that its purpose is to discuss particular methods such as normal and *t*. In such a case, an omission of mention of parametric methods may appear defensible.

We therefore formulate completeness as *completeness <u>relative</u> to the parameters the narrative introduces to describe the decision rule.* Table 1 illustrates this approach: The table requires consideration of three issues, i) whether the underlying population is normal, ii) whether the population variance is known, and iii) whether the sample size is large. Consequently, *completeness* of a decision rule requires that the decision rule lays down what happens in each combination of these three requirements. It immediately follows that, *based* on its choice of parameters, the narrative explicitly requires a discussion of the non-parametric approach: when the sample size is small, and the population is not normal.

It is interesting that some undergraduate textbooks or courses, *by design*, omit parametric methods from introductory statistics courses. Fortunately, some recent textbooks remedy this problem by providing appropriate material [18].

The author's practice, when teaching from a curriculum not requiring teaching parametric methods, is to point out that the correct response to a question where the population is not normal and the sample size is small is, "This question cannot be answered by the methods of this course."

It is noteworthy, that even though students are explicitly told that at least one question on examinations cannot be approached through the normal and *t* distributions, weaker students, typically get these questions incorrect; they approach these problems using the normal and *t* distributions without first applying the decision rules presented in Table 1 to ascertain what method should be used. This anecdote points to the challenge in teaching material based on multiple parameters: such material lends itself to avoidance of certain cases by weaker students.

In summary, the PINF has a variety of nice attributes:
✓ It is complete
✓ It is brief and compact
✓ It is readable
✓ It does not contain any redundant information.

The author's experience with the PINF is that it is superior for instruction, retention, and learning. While flowcharts are also complete [5], the flowchart is somewhat overwhelming, lacks compactness, and cannot easily be learned. Flowcharts are typically only useful when considering one path in the flowchart. Additionally, the author has found the PINF approach useful for remediation. The author explains to a weaker student who confuses rules that their problem is not mathematical but logical. Many of these students have never seen a multiple-parameter rule. Such a remediation approach can help some of the weaker students.

4. RELATIVE PRONOUNS

This article includes examples from disparate disciplines, Mathematics and English, in order to emphasize the cybernetic nature of the material presented. Cybernetics is intrinsically multi-disciplinary, since in its essence, it refers to the flow and organization of information *independent of content* [21]. The cybernetic flavor of this article helps the reader focus on the abstract Boolean nature of the PINF format advocated.

4.1 Rules Governing the Five Definite Relative Pronouns: The topic of *definite relative pronouns* was selected for this paper's second example for a variety of reasons. i) There are only five definite relative pronouns in English: *that, who, whom, which, whose.* (Occasionally, but not always, *where, when, why* can also function as relative pronouns. There are also indefinite relative pronouns: *whoever, whomever.*) ii) Despite the small number of definite relative pronouns, the theory is rich and nuanced; even experienced native speakers can get caught in errors.

4.2 The Rules and Their PINF: Table 3 compactly presents i) the rules, ii) the parameters driving them, and iii) illustrative examples. The following explanatory comments clarify further:

- For purely formatting reasons the following abbreviations are used in order to fit the table to the page: *sub* refers to a case where the relative pronoun is the *subject* of the clause it heads; *obj* refers to a case where the relative pronoun is the *direct object* of the verb of the underlying sentence; *prep.* refers to a relative pronoun in a *prepositional* clause. Similarly, the *who* and *who(m)* rules in the row for *object with preposition at end*, are combined (again for purely formatting reasons).
- Table 1 listed questions in the column headers and responses to these questions, yes or *no*, in the cells. Because of the complexity of the relative pronoun rules a slightly different format is used here. For example, Row 1 presents two rules: *if the subject of the sentence is a person*, then the rule requires using *who* or *that*, while *if the subject of the sentence is an impersonal object* the rule requires using *which* or *that*. This reflects the fact that the question or issue, *what is the type of the subject?* is answered with the two possibilities *people* or *objects*.
- Illustrative examples follow each rule. For example, the sentences *I praised the student who got A, I praised the student that got A,* and *I praised the student getting an A* are all examples where the relative pronoun (*who, that, or blank*) is the subject of the sentence clause it is connected with (*The student got A*).

Case	For People	Examples (people)	For Things	Examples (Things)
Sub.	WHO THAT	*1)I praised the student WHO earned A* 2) I praised the student THAT earned A	WHICH THAT	*1)I praised the grades WHICH were all A* 2) I praised the grades THAT were all A
Obj	WHO THAT BLANK	*1)I praised the student WHO got A* 2) I praised the student THAT got A *3)I praised the student___getting A*	WHICH THAT BLANK	*1)The grades WHICH I praised were A-s* 2) The grades THAT I praised were A-s *3) The grades _____ I praised were A-s*
Obj., Prep. at end	WHOM WHO THAT BLANK	*1-2) This is the student WHO(M) I gave A TO* 3) This is the student THAT I gave A TO *4) This is the student ___ I gave A TO*	WHICH THAT BLANK	*1)This is the report WHICH I recorded A-s on* 2) This is the report THAT I recorded A-s on *3) This is the report ____ I recorded A-s on*
Obj., Prep. at begin	WHOM	*This is the student TO WHOM I gave A*	WHICH	*This is the report ON WHICH I placed A*
Possess-ive	WHOSE	*This is the student WHOSE grade is A*	WHOSE	*This is the report WHOSE grades are all A-s*

Table 3: The decision table, with examples, for definite relative pronouns.

- As can be seen, the rules are similar for some columns (for example, *who, that, and blank (pronoun absence)* occur with variations in several rows). Understanding such associations facilitates student learning, retention, and avoidance of error. As indicated in Section 1, *table format,* is superior to the *list format* in communicating such associations.

4.3 Comparisons with Alternative Rule Presentations: As done for the statistics rule presented in Section 3, a Google search, using the search phrase "the rules governing relative pronouns" was made in March 2023 leading to a variety of online sources, from universities and online tutorials [4, 6, 14, 15, 16]. These, sources, supplemented with one textbook, [1], were compared and contrasted. Rather than review each source, certain general tendencies are summarized.

- Use of Tables vs. lists: Both list and table formats are used.
- Incompleteness: Certain minutiae in the table are overlooked by some sources. For example: i) the use of *whom* and *who* for the case of *object of a verb* is overlooked in some sources; contrastively, some sources correctly note that use of *whom vs. who* corresponds to proper usage in formal vs. conversational English; ii) The *blank* rule (permissibility of absence of a relative pronoun) is sometimes omitted; and iii) The dependency for the object-preposition case on whether the preposition is placed prior to the relative pronoun or after it is also sometimes omitted. In general, rules that require introduction of extra parameters tend to be omitted.
- Integrated Rules: Table 2 assumes student familiarity with identifying the relative pronoun as being a subject, object, or indirect object (object of a preposition). Contrastively, some sources integrate the rules for syntactical recognition with the relative pronoun usage. Integration of modules, particularly, when there is a review and reference to a prior module, is good pedagogical practice and facilitates avoidance of silo effects where each rule stands by itself without its relationship to other rules.
- Integration Presentation of Examples and Rules: Table 2 integrates the presentation of rules and examples. This is a superior pedagogic practice. Some sources adopt this practice; other sources give preference to spreading out the statement of rules by studying examples after each individual rule statement.
- Additional Elaborations: One source explicitly notes that pronouns that refer to animals are treated the same way as pronouns referring to inanimate things. It is a matter of taste, and dependent on the target student population, to what extent an instructor wishes to emphasize these subtleties.

5. CONCLUSION

This paper has presented best practices of teaching delivery when a multi-parameter rule is present. These best practices are formulated with the mathematical logic concept of the Prime Implicant Normal Form, PINF. Examples using PINF are presented from disparate disciplines and compared to a variety of alternative presentations currently being used. The comparative analysis shows that the PINF technique: i) facilitates completeness of rules without omissions, ii) avoids narrative styles that are not conducive to presenting multi-parameter rules, iii) provides clearer logical statements, and facilitates student learning and retention, and iv) is easy to learn, master, and apply.

It is hoped that this modest presentation will suffice to inspire instructors to use this technique in their courses.

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